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Space-time symplectic extension

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Abstract

It is conjectured that in the origin of space-time there lies a symplectic rather than metric structure. The symplectic symmetry $Sp(2l, C)$, $l \geq 1$ instead of the pseudo-orthogonal one $SO(1, d-1)$, $d \geq 4$ is proposed as the space-time local structure group. A discrete sequence of the metric space-times of the fixed dimensionalities $d = (2l)^2$ and signatures, with $l(2l-1)$ time-like and $l(2l+1)$ space-like directions, defined over the set of the Hermitian second-rank spin-tensors is considered as an alternative to the pseudo-Euclidean extra dimensional space-times. The basic concepts of the symplectic framework are developed in general, and the ordinary and next-to-ordinary space-time cases with $l = 1, 2$, respectively, are elaborated in more detail. In particular, the scheme provides the rationale for the four-dimensionality and $1 + 3$ signature of the ordinary space-time.

1 Introduction

At present, the ordinary space-time is postulated to be locally the Minkowski space, i.e., the pseudo-Euclidean space of the dimensionality $d = 4$ with the Lorentz group $SO(1,3)$ as the local symmetry group. Nevertheless, the spinor analysis in the Minkowski space heavily relies on the isomorphism for the proper noncompact groups $SO(1,3) \simeq SL(2,C)/Z_2$, as well as that $SO(3) \simeq SU(2)/Z_2$ for their maximal compact subgroups (see, e.g., [1]). Moreover, the whole relativistic field theory in four space-time dimensions can equivalently be formulated (and in a sense it is even preferable) entirely in the framework of spinors of the $SL(2,C)$ group [2]. In this approach, to a space-time point there corresponds a Hermitian tensor of the second rank.

From this point of view, a description of the ordinary space-time by means of the real four-vectors of the $SO(1,3)$ group, rather than by the Hermitian tensors of $SL(2,C)$, is nothing but the (historically settled) tradition of the space-time parametrization. Nevertheless, right this parametrization underlies the proposed and widely discussed space-time extensions into the (locally) pseudo-Euclidean spaces of the larger dimensionalities $d > 4$ in the Kaluza-Klein fashion (see, e.g., [3]). These extensions assume the embedding of the local symmetry groups as $SO(1,3) \subset SO(1,d-1)$. The pseudo-Euclidean extensions play the crucial role in the attempts to construct a unified theory of all the interactions including gravity [4].

In what follows we stick to the viewpoint that spinors are more fundamental objects than vectors. Thus the space-time structure group with spinors as defining representations, i.e. the complex symplectic group $Sp(2,C)$, is considered to be more appropriate than the pseudo-orthogonal group $SO(1,3)$ with vectors as defining representations and spinors just as a kind of artefact. In other words, we assume that the symplectic structure of the space-time has a deeper physical origin than the metric one though both approaches, symplectic and pseudo-orthogonal, are formally equivalent at an effective level in the ordinary space-time. Then in searching for the space-time extra dimensional extensions, a natural step would be to look for the extensions in the symplectic framework with the structure group $Sp(2l,C)$, $l > 1$. The reason is that the descriptions equivalent at $l = 1$ and $d = 4$ can result in principally different extensions at $l > 1$ and $d > 4$. This is the problem dealt with in the present paper. We develop the basic concepts of the general symplectic framework and elaborate in more detail the ordinary and next-to-ordinary space-time cases with $l = 1, 2$, respectively.[1]

[1] An early version of the study can be found in [5].

2 Structure group

It is assumed that an underlying physics described effectively by a local symmetry (structure group) constitutes the basis for the local properties of the space-time, i.e., for its dimensionality and signature. Hence, to find possible types of the space-time extensions it is necessary first of all to find out all the structure groups isomorphic each other at $d = 4$. In addition to the well-known isomorphism of the real and complex groups $SO(1,3) \simeq SL(2,C)/Z_2$ relevant to the ordinary space-time, there exist the following isomorphisms (up to Z_2) for the proper complex Lie groups: $SL(2,C) \simeq SO(3,C) \simeq Sp(2,C)$ and, respectively, for their maximal compact (real) subgroups $SU(2) \simeq SO(3) \simeq Sp(2)$. In other terms these isomorphisms look like $A_1 \simeq B_1 \simeq C_1$, where the groups considered are the first ones from the complex Cartan series: $A_l = SL(l+1,C)$, $B_l = SO(2l+1,C)$, $C_l = Sp(2l,C)$ and similarly for their maximal compact subgroups $SU(l+1)$, $SO(2l+1)$, $Sp(2l)$ (see, e.g., [6]). Here $l \geq 1$ means the rank of the corresponding Lie algebras. It is equal to the half-rank of the proper noncompact Lie groups and coincides with the rank of their maximal compact subgroups. As the structure groups, all the groups from the above series result in the (locally) isomorphic descriptions at $l = 1$. Therefore at $l > 1$, the extended structure groups may a priori be looked for in each of the series with properly extended spinor space. But the physical requirement for the existence of an invariant bilinear product in the extended spinor space restricts the admissible types of extension.

Namely, for all the complex groups the complex conjugate fundamental representations $\bar{\psi}$ are not equivalent to the representations ψ themselves. Besides, for all the complex series there is no invariant tensor in the spinor space which would match a spinor representation and its complex conjugate. Hence, the invariant bilinear product of Grassmann fields in the form $\psi\psi$ (and $\bar{\psi}\bar{\psi}$) is the only possible one (if any). The latter is admissible just for the symplectic series C_l. This is due to the fact that, by definition, there exists in this case the invariant (antisymmetric) second-rank tensor. It is to be noted, that the spinor representations of the orthogonal groups B_l are realized by the embedding of the latter ones into the symplectic groups $C_{2^{l-1}}$ over the 2^l-dimensional spinor space. Only at $l = 1, 2$ there take place the isomorphisms $B_l \simeq C_l$. The spinors being assumed to be more fundamental objects than vectors, it is natural to consider directly the symplectic groups which are self-sufficient for spinors, instead of the pseudo-orthogonal ones which inevitably should appeal to symplectic groups for justification of the spinor representations.

Just the existence of the alternating second-rank tensor in the $SL(2,C)$ group is, in essence, the *raison d'etre* for the spinor analysis in four space-time

dimensions being based traditionally on this group. The symmetry structure which provides the alternating tensor and, as a result, the invariant inner product for spinors, proves to be crucial for the whole physical theory. But this structure survives in $Sp(2l,C)$ and is absent in $SL(l+1,C)$ at $l>1$. This is why namely the first groups, and not the second ones, are to be considered as the structure groups of the extended space-time. Therefore, while constructing extra dimensional space-times we retain symplectic structure, i.e., consider extensions in the series C_l.

To summarize: two alternative ways of the space-time extension can be pictured schematically as

$$\begin{array}{ccc} SO(1,3) & \simeq & Sp(2,C) \\ \downarrow & & \downarrow \\ SO(1,d-1) & \not\simeq & Sp(2l,C)\,. \end{array} \tag{1}$$

The first, commonly adopted way of extension, corresponds to the real structure groups while the second one relies on the complex groups. The scheme shows that the isomorphism of the real and complex groups, valid at $d=4$ and $l=1$, is no longer fulfilled at $d>4$ and $l>1$. In the first way of extension the local metric properties of the space-time (i.e., dimensionality and signature) are put in ab initio. In the second way, these properties should not be considered as the primary ones but, instead, they have to emerge as a manifestation of the inherent symplectic structure.

3 Sp(2l,C)

Let ψ_A and $\bar\psi^{\bar A}\equiv(\psi_A)^*$, as well as their respective duals ψ^A and $\bar\psi_{\bar A}\equiv(\psi^A)^*$, A, $\bar A=1,\ldots,n$ $(n=2l)$ are the spinor representaions of $Sp(2l,C)$. It is well known that there exist in the spinor space the nondegenerate invariant second-rank spin-tensors $\epsilon_{AB}=-\epsilon_{BA}$ and $\epsilon^{AB}=-\epsilon^{BA}$ such that $\epsilon_{AC}\epsilon^{CB}={\delta_A}^B$, with ${\delta_A}^B$ being the Kroneker symbol (and similarly for $\epsilon_{\bar A\bar B}\equiv(\epsilon^{BA})^*$ and $\epsilon^{\bar A\bar B}\equiv(\epsilon_{BA})^*$). Owing to these invariant tensors the spinor indices of the upper and lower positions are pairwise equivalent ($\psi_A\sim\psi^A$ and $\bar\psi_{\bar A}\sim\bar\psi^{\bar A}$), so that there are left just two inequivalent spinor representaions (generically, ψ and $\bar\psi$). Let us call ψ and $\bar\psi$ the spinors of the first and the second kind, respectively, and similarly for the corresponding indices A and $\bar A$.[2]

[2]Note that both the type and position of the indices are changed under complex conjugation, contrary the traditional definition of the dotted indices for $SL(2,C)$ without the position change: $(\psi_A)^*\equiv\psi^*_{\dot A}$, etc. The advantage of the definition adopted in the present paper is that relative to the maximal compact subgroup $Sp(2l)$, the two types of indices A and $\bar A$ in the same position are completely indistinguishable, while the similar A and $\dot A$ would enjoy this property only after the implicit position change for $\dot A$.

Let us put in correspondence to an event point P a second rank spin-tensor $X_A{}^{\bar B}(P)$, which is Hermitian, i.e., $X_A{}^{\bar B} = (X_B{}^{\bar A})^* \equiv \bar X^{\bar B}{}_A$, or in other terms $X^{A\bar B} = (X_{B\bar A})^*$. One can define the quadratic scalar product as

$$\mathrm{tr}\, X\bar X \equiv X_A{}^{\bar B}\bar X_{\bar B}{}^A = X_A{}^{\bar B}X^A{}_{\bar B} = -X_{A\bar B}X^{A\bar B} = -X_{A\bar B}(X_{B\bar A})^* \,, \tag{2}$$

the last equality being due to the Hermiticity of X. Clearly, $\mathrm{tr}\, X\bar X$ is real though not sign definite. Besides, the spin-tensor $X\bar X$ is antisymmetric, $(X\bar X)_{AB} = -(X\bar X)_{BA}$, and hence it can be decomposed into the trace relative to ϵ and a traceless part. Under $S \in Sp(2l, C)$ one has in compact notations:

$$\begin{aligned} X &\to SXS^\dagger \,, \\ \bar X &\to S^{\dagger -1}\bar X S^{-1} \,, \end{aligned} \tag{3}$$

so that $X\bar X \to SX\bar X S^{-1}$ and $\mathrm{tr}\, X\bar X$ is invariant, indeed. In fact, the invariant (2) is at $l > 1$ just the first one in a series of independent invariants $\mathrm{tr}\,(X\bar X)^k$, $k = 1, \ldots, l$. By definition, set $\{X\}$ endowed with the structure group $Sp(2l, C)$ and the interval between points X_1 and X_1 defined as $\mathrm{tr}\,(X_1 - X_2)(\bar X_1 - \bar X_2)$ constitutes the symplectic space-time. The non-compact transformations from the $Sp(2l, C)$ are counterparts of the Lorentz boosts in the ordinary space-time, while transformations from the compact subgroup $Sp(2l) = Sp(2l, C) \cap SU(2l)$ correspond to rotations. With account for translations $X_A{}^{\bar B} \to X_A{}^{\bar B} + \Xi_A{}^{\bar B}$, where $\Xi_A{}^{\bar B}$ is an arbitrary constant Hermitian spin-tensor, the whole theory in the flat symplectic space-time should be invariant under the inhomogeneous symplectic group.

Let us now fix for a while the extended boosts and restrict ourselves by the extended rotations, i.e., by the maximal compact subgroup $Sp(2l)$. Relative to the latter, the indices of the first and the second types are indistinguishable in their transformation properties ($\psi_A \sim \bar\psi_{\bar A}$), and one can temporarily label $X_{A\bar B}$ in this case as X_{XY}, where $X, Y, \ldots = 1, \ldots, n$ generically mean spinor indices irrespective of their kind. Hence, while restricting by the compact subgroup one can reduce the tensor X_{XY} into two irreducible parts, symmetric and antisymmetric ones: $X_{XY} = \sum_\pm (X_\pm)_{XY}$, where $(X_\pm)_{XY} = \pm(X_\pm)_{YX}$ have $d_\pm = n(n \pm 1)/2$ dimensions, respectively. One gets from (2) the following decomposition for the scalar product:

$$\mathrm{tr}\, X\bar X = \sum_\pm (\mp 1)(X_\pm)_{XY}[(X_\pm)_{XY}]^* \,. \tag{4}$$

At $l > 1$, one can further reduce spin-tensor X_- into the trace $X_-^{(0)}$ relative to ϵ and a traceless part $X_-^{(1)}$ as $(X_-)_{XY} = 1/\sqrt{n}\, X_-^{(0)}\epsilon_{XY} + (X_-^{(1)})_{XY}$ so that

$$\mathrm{tr}\, X\bar X = X_-^{(0)2} + (X_-^{(1)})_{XY}[(X_-^{(1)})_{XY}]^* - (X_+)_{XY}[(X_+)_{XY}]^* \,. \tag{5}$$

As a result, the whole extended space-time can be decomposed with respect to the rotation group into three irreducible subspaces of the 1, $(n-2)(n+1)/2$ and $n(n+1)/2$ dimensions. According to their signature and transformation properties, the first two subspaces correspond to the time extra dimensions, the rotationally invariant and non-invariant ones, while the third subspace corresponds to the spatial extra dimensions. It is to be noted that the number of components in the extended space, and hence that in the spatial momentum, is equal to the number of the noncompact transformations (boosts). Thus, for a massive particle there exist a rest frame with zero spatial momentum. In the case $n = 2$ there is a unique antisymmetric tensor $(X_-)_{XY} \sim \epsilon_{XY}$, so that the non-invariant time subspace is empty.

Of course, the particular decomposition of X into two parts $X_\pm$ is noncovariant with respect to the whole $Sp(2l, C)$ and depends on the boosts. Nevertheless, the decomposition being valid at any boost, the numbers of the positive and negative components in $\mathrm{tr} X\bar{X}$ is invariant under the whole $Sp(2l, C)$. In other words, the metric signature of the symplectic space-time

$$\sigma_d = (\underbrace{+1, \dots}_{d-}; \underbrace{-1, \dots}_{d_+}) \tag{6}$$

is invariant. Hence, at $n = 2l > 2$ the structure group $Sp(2l, C)$ of the n-th rank and the $n(n-1)$-th order, acting on the Hermitian second-rank spin-tensors with $d = n^2$ components, is just a restricted subgroup of the embedding pseudo-orthogonal group $SO(d_-, d_+)$, of the rank $n^2/2$ and the order $n^2(n^2-1)/2$, acting on the pseudo-Euclidean space of the dimensionality $d = n^2$. What distinguishes $Sp(2l, C)$ from $SO(d_-, d_+)$, is the total set of independent invariants $\mathrm{tr}(X\bar{X})^k$, $k = 1, \dots, l$. The isomorphism between the groups is achieved only at $l = 1$, i.e., for the ordinary space-time $d = 4$ where there is just one invariant $\mathrm{tr} X\bar{X}$.

It should be stressed that in the approach under consideration, neither the discrete set of dimensionalities, $d = (2l)^2$, of the extended space-time, nor its signature, nor the existence of the rotationally invariant one-dimensional time subspace are postulated ab initio. Rather, they are the immediate consequences of the underlying symplectic structure. In particular, the latter seems to provide the unique rationale for the four-dimensionality of the ordinary space-time, as well as for its signature $(+ - --)$. Namely, these properties directly reflect the existence of one antisymmetric and three symmetric second-rank Hermitian spin-tensors at $l = 1$. The set of such tensors, in its turn, is the lowest admissible Hermitian space to accommodate the symplectic structure, the case $l = 0$ being trivial ($d = 0$). On the other hand, right the existence of the one-dimensional time subspace allows one to (partially) order the events at any fixed boosts, which serves as a basis for the causality description. Hence, the latter may ultimately be attributed

to the underlying symplectic structure, too. At $l > 1$, because of the extra times being mixed via boosts with the one-dimensional time, the causality should approximately be valid only at small boosts.

4 C, P, T

Let us charge double the spinor space, i.e., for each ψ_A, $(\psi_A)^\dagger \equiv \bar{\psi}^{\bar{A}}$ introduce two copies $\psi_A^\pm$, $(\psi_A^\pm)^\dagger \equiv (\bar{\psi}^\mp)^{\bar{A}}$, with $\pm$ being the "charge" sign.[3] In analogy to the ordinary case of $SL(2,C)$ [1], one can define the following discrete symmetries:

$$\begin{aligned} C &: \psi_A^\pm \to \psi_A^\mp\,, \\ P &: \psi_A^\pm \to (\psi_A^\mp)^\dagger \equiv (\overline{\psi}^\pm)^{\bar{A}}\,, \\ T &: \psi_A^\pm \to (\psi_A^\pm)^\dagger \equiv (\overline{\psi}^\mp)^{\bar{A}}\,, \end{aligned} \tag{7}$$

and hence $CPT : \psi_A^\pm \to \psi_A^\pm$ (all up to the phase factors). Under CPT invariance, only two of the discrete operations (7) are independent ones. Without charge doubling, just one combination $CP \equiv T : \psi_A \to \bar{\psi}^{\bar{A}}$ survives.

Now, let us introduce the Hermitian spin-tensor current $J = J^\dagger$ as follows

$$J_A{}^{\bar{B}} \equiv \sum_\pm (\pm 1)\psi_A^\pm (\psi_B^\pm)^\dagger = \sum_\pm (\pm 1)\psi_A^\pm (\overline{\psi}^\mp)^{\bar{B}}\,. \tag{8}$$

(ψ's are the Grassmann fields). Under (7) the current $J_A{}^{\bar{B}}$ transforms as follows

$$\begin{aligned} C &: J_A{}^{\bar{B}} \to -J_A{}^{\bar{B}}\,, \\ P &: J_A{}^{\bar{B}} \to -J_B{}^{\bar{A}}\,, \\ T &: J_A{}^{\bar{B}} \to J_B{}^{\bar{A}}\,. \end{aligned} \tag{9}$$

Fixing boosts and decomposing current $J_{A\bar{B}}$ into the symmetric and antisymmetric parts, $J_{XY} = \sum_\pm (J_\pm)_{XY}$, one gets from (9):

$$\begin{aligned} C &: (J_\pm)_{XY} \to -(J_\pm)_{XY}\,, \\ P &: (J_\pm)_{XY} \to \mp(J_\pm)^{XY}\,, \\ T &: (J_\pm)_{XY} \to \pm(J_\pm)^{XY}\,. \end{aligned} \tag{10}$$

This is in complete agreement with the signature association for the symmetric (antisymmetric) part of the Hermitian spin-tensor X as the extended spatial (time) components.

[3] We use here a dagger sign for complex conjugation to show that the Grassmann fields should undergo the change of the order in their products.

5 l = 1

The noncompact group $Sp(2l,C)$ has $n(n{+}1)$ generators $M_{AB} = (L_{AB}, K_{AB})$, $A, B = 1, \ldots, n$ ($n = 2l$), so that $L_{AB} = L_{BA}$ and similarly for K_{AB}. The generators L_{AB} are Hermitian and correspond to the extended rotations, whereas those K_{AB} are anti-Hermitian and correspond to the extended boosts. In the space of the first-kind spinors ψ_A these generators can be represented as $(\sigma_{AB}, i\sigma_{AB})$ with $(\sigma_{AB})_{CD} = 1/2(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})$, so that $\sigma_{AB} = \sigma_{BA}$ and $(\sigma_{AB})_{CD} = (\sigma_{AB})_{DC}$, $(\sigma_{AB})_C{}^C = 0$. Similar expressions hold true in the space of the second-kind spinors $\bar\psi_{\bar A}$. In these terms, a canonical formalism can be developed at arbitrary $l \geq 1$.

However, in the simplest case $l = 1$ corresponding to the ordinary four-dimensional space-time, there exists the isomorphism $B_1 \simeq C_1$ (or $SO(3,C) \simeq Sp(2,C)/Z_2$). Due to this property, the structure of $Sp(2,C)$ can be brought to the form, though equivalent mathematically, more familiar physically.[4] Namely, let us introduce for the $SO(3,C)$ group the double set of the Pauli matrices, $(\sigma_i)_A{}^{\bar B}$ and $(\overline\sigma_i)_{\bar A}{}^B$, $i = 1,2,3$. They should satisfy the anticommutation relations: $\sigma_i\overline\sigma_j + \sigma_j\overline\sigma_i = 2\delta_{ij}\sigma_0$ and $\overline\sigma_i\sigma_j + \overline\sigma_j\sigma_i = 2\delta_{ij}\overline\sigma_0$, where $(\sigma_0)_A{}^B \equiv \delta_A{}^B$, $(\overline\sigma_0)_{\bar A}{}^{\bar B} \equiv \delta_{\bar A}{}^{\bar B}$ are the Kroneker symbols and δ_{ij} is the metric tensor of $SO(3,C)$. Among these matrices, σ_0 and $\overline\sigma_0$ are the only independent ones which can be chosen antisymmetric, $(\sigma_0)_{AB} \equiv \epsilon_{AB}$ and $(\overline\sigma_0)_{\bar A\bar B} \equiv \epsilon_{\bar A\bar B}$. On the other hand, with respect to the maximal compact subgroup $SO(3)$, all the matrices σ_i, $\overline\sigma_i$ can be chosen both Hermitian and symmetric as $(\sigma_i)_X{}^Y = [(\sigma_i)_Y{}^X]^*$ and $(\sigma_i)_{XY} = (\sigma_i)_{YX}$ (and the same for $\overline\sigma_i$). The matrices $\sigma_{ij} \equiv -i/2\,(\sigma_i\overline\sigma_j - \sigma_j\overline\sigma_i)$, such that $\sigma_{ij} = -\sigma_{ji}$ and $(\sigma_{ij})_{AB} = (\sigma_{ij})_{BA}$ (and similarly for $(\overline\sigma_{ij})_{\bar A\bar B} \equiv i/2\,(\overline\sigma_i\sigma_j - \overline\sigma_j\sigma_i)_{\bar A\bar B}$), are not linearly independent from σ_i. They can be brought to the form $(\sigma_{ij})_{XY} = \epsilon_{ijk}\,(\sigma_k)_{XY}$, with ϵ_{ijk} being the Levi-Civita $SO(3,C)$ symbol.

The matrices $(\sigma_{ij}, i\sigma_{ij})$ can be identified as the generators $M_{ij} = (L_{ij}, K_{ij})$ of the noncompact $SO(3,C)$ group in the space of the first-kind spinors. Respectively, in the space of the second-kind spinors they are $(-\overline\sigma_{ij}, i\overline\sigma_{ij})$. The generators L_{ij} of the maximal compact subgroup $SO(3) \simeq Sp(2)/Z_2$ correspond to rotations, while those K_{ij} of the noncompact transformations describe Lorentz boosts. Relative to $SO(3)$ one has $\bar\sigma_0 = \sigma_0$, $\bar\sigma_i = \sigma_i$ and $\bar\sigma_{ij} = -\sigma_{ij}$. When restricted by the maximal compact subgroup $SO(3)$, the Hermitian second-rank spin-tensor may be decomposed in the complete set of the Hermitian matrices (σ_0, σ_{ij}) with the real coefficients: $X = 1/\sqrt{2}\,(x_0\sigma_0 + 1/2\,x_{ij}\sigma_{ij})$, so that $\mathrm{tr}X\bar X = x_0^2 - 1/2\,x_{ij}^2$. With identification $x_{ij} \equiv \epsilon_{ijk}x_k$ one

[4] We use here the complex group $SO(3,C)$ instead of the real one $SO(1,3)$ to show the close similarity with the next case $l = 2$ where there is no real structure group. Because of the complexity of $SO(3,C)$ one should distinguish vectors and their complex conjugate, the latter ones being omitted for simplicity in what follows. The same remains true for the $SO(5,C)$ case corresponding to $l = 2$.

gets as usually $\mathrm{tr}X\bar{X} = x_0^2 - x_i^2$. Both the time and spatial representations being irreducible under $SO(3)$, there takes place the usual decomposition $\underline{4} = \underline{1} \oplus \underline{3}$ relative to the embedding $SO(3,C) \supset SO(3)$.

6 l = 2

This case corresponds to the next-to-ordinary space-time symplectic extension. Similarly to the case $l = 1$, there takes place the isomorphism $B_2 \simeq C_2$, or $SO(5,C) \simeq Sp(4,C)/Z_2$. Cases $l = 1, 2$ are the only ones when the structure of the symplectic group gets simplified in terms of the complex orthogonal groups. The double set of Clifford matrices $(\Sigma_I)_A{}^{\bar{B}}$ and $(\overline{\Sigma}_I)_{\bar{A}}{}^B$, $I = 1,\ldots,5$ satisfies $\Sigma_I\overline{\Sigma}_J + \Sigma_J\overline{\Sigma}_I = 2\delta_{IJ}\Sigma_0$ and $\overline{\Sigma}_I\Sigma_J + \overline{\Sigma}_J\Sigma_I = 2\delta_{IJ}\overline{\Sigma}_0$, where $(\Sigma_0)_A{}^B \equiv \delta_A{}^B$, $(\overline{\Sigma}_0)_{\bar{A}}{}^{\bar{B}} \equiv \delta_{\bar{A}}{}^{\bar{B}}$ are the Kroneker symbols and δ_{IJ} is the metric tensor of $SO(5,C)$. Relative to the maximal compact subgroup $SO(5)$ they may be chosen Hermitian, $(\Sigma_I)_X{}^Y = [(\Sigma_I)_Y{}^X]^*$, but antisymmetric $(\Sigma_I)_{XY} = -(\Sigma_I)_{YX}$ (and similarly for $\overline{\Sigma}_I$), like $(\Sigma_0)_{AB} = \epsilon_{AB}$ and $(\overline{\Sigma}_0)_{\bar{A}\bar{B}} = \epsilon_{\bar{A}\bar{B}}$. One can also require that $(\Sigma_I)_X{}^X = 0$. Therefore, under restriction by $SO(5)$, six matrices Σ_0, Σ_I provide the complete independent set for the antisymmetric matrices in the four-dimensional spinor space. After introducing matrices $\Sigma_{IJ} = -i/2(\Sigma_I\overline{\Sigma}_J - \Sigma_J\overline{\Sigma}_I)$, so that $\Sigma_{IJ} = -\Sigma_{JI}$, one gets the symmetry condition for them: $(\Sigma_{IJ})_{AB} = (\Sigma_{IJ})_{BA}$ (and similarly for $(\overline{\Sigma}_{IJ})_{\bar{A}\bar{B}} = i/2(\overline{\Sigma}_I\Sigma_J - \overline{\Sigma}_J\Sigma_I)_{\bar{A}\bar{B}}$). Hence, ten matrices Σ_{IJ} (or $\overline{\Sigma}_{IJ}$) make up the complete set for the symmetric matrices in the spinor space. Under $SO(5)$ one has $\overline{\Sigma}_0 = \Sigma_0$, $\overline{\Sigma}_I = \Sigma_I$ and $\overline{\Sigma}_{IJ} = -\Sigma_{IJ}$.

With respect to $SO(5)$ the Hermitian second-rank spin-tensor X may be decomposed in the complete set of matrices Σ_0, Σ_I and Σ_{IJ} with the real coefficients: $X = 1/2\,(x_0\Sigma_0 + x_I\Sigma_I + 1/2\,x_{IJ}\Sigma_{IJ})$. In these terms one gets

$$\mathrm{tr}X\bar{X} = x_0^2 + x_I^2 - \frac{1}{2}x_{IJ}^2\,. \tag{11}$$

There is one more independent invariant combination of x_0, x_I and x_{IJ} stemming from the invariant $\mathrm{tr}(X\bar{X})^2$. Relative to the embedding $SO(5,C) \supset SO(5)$ one has the following decomposition in the irreducible representations:

$$\underline{16} = \underline{1} \oplus \underline{5} \oplus \underline{10}\,. \tag{12}$$

Under the discrete transformations (7) one gets

$$\begin{aligned} P &: \; x_0 \to x_0,\; x_I \to x_I,\; x_{IJ} \to -x_{IJ}\,, \\ T &: \; x_0 \to -x_0,\; x_I \to -x_I,\; x_{IJ} \to x_{IJ}\,. \end{aligned} \tag{13}$$

This means that from the point of view of $SO(5)$, x_I is the axial vector

whereas x_{IJ} is the pseudo-tensor (a counterpart of $x_{ij} = \epsilon_{ijk}x_k$ in three spatial dimensions). The matrices $(\Sigma_{IJ},\ i\Sigma_{IJ})$ or $(-\overline{\Sigma}_{IJ},\ i\overline{\Sigma}_{IJ})$ represent the $SO(5,C)$ generators $M_{IJ} = (L_{IJ}, K_{IJ})$ in the spaces of the spinors, respectively, of the first and the second kinds. A particular expression for the matrices Σ_I, Σ_{IJ} in terms of σ_0, σ_i depends on the fashion of the embedding $SO(3,C) \subset SO(5,C)$.

The rank of the algebra C_2 being $l = 2$, an arbitrary irreducible representation of the noncompact group $Sp(4,C)$ is uniquely characterized by two complex Casimir operators I_2 and I_4, respectively, of the second and the forth order, i.e., by four real quantum numbers. Otherwise, an irreducible representation of $Sp(4,C)$ can be described by the mixed spin-tensor $\Psi^{\bar{B}_1\cdots}_{A_1\cdots}$ of a proper rank. This spin-tensor should be traceless in any pair of the indices of the same kind, and its symmetry in each kind of the indices should correspond to a two-row Young tableau. In fact, there exists the completely antisymmetric invariant tensor of the fourth rank $\epsilon_{A_1A_2A_3A_4} \equiv \epsilon_{A_1A_2}\epsilon_{A_3A_4} - \epsilon_{A_1A_3}\epsilon_{A_2A_4} + \epsilon_{A_1A_4}\epsilon_{A_2A_3}$ which corresponds to the embedding $SL(4,C) \supset Sp(4,C)$ (and similarly for $\epsilon_{\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4}$). By means of these invariant tensors, three indices of the same kind with antisymmetry are equivalent to one index, whereas four indices with antisymmetry can be omitted altogether. Hence, antisymmetry is possible in no more than pairs of indices of the same kind. Therefore, an irreducible representaion of $Sp(4,C)$ may unambiguously be characterized by a set of four integers $(r_1, r_2; \bar{r}_1, \bar{r}_2)$, $r_1 \geq r_2 \geq 0$ and $\bar{r}_1 \geq \bar{r}_2 \geq 0$. Here $r_{1,2}$ (respectively, $\bar{r}_{1,2}$) are the numbers of boxes in the first or the second rows of the proper Young tableau. The rank of the maximal compact subgroup $SO(5) \simeq Sp(4)/Z_2$ (the rotation group) being equal to $l = 2$, a state in a representaton is additionally characterized under fixed boosts by two additive quantum numbers, namely, the eigenvalues of the mutually commuting momentum components of L_{IJ} in two different planes, say, L_{12} and L_{45}. Note, that in the $Sp(2,C)$ case the Young tableaux are at most one-rowed, and an irreducible representation is characterized by a pair of integers $(r; \bar{r})$, with the complex dimensionality of the representation being $(r+1)(\bar{r}+1)$. In this case, there remains just one diagonal component of the total angular momentum, say, $L_{12} \equiv L_3$.

7 l → 1 reduction

The ultimate of the dimensionality in the given approach is the discrete number $l = 1, 2, \ldots$ corresponding to the dimensionality $n = 2l$ of the spinor space. The dimensionality $d = (2l)^2$ of the space-time appears just as a secondary quantity. In reality, the extended space-time with $l > 1$ should compactify to the ordinary one with $l = 1$ by means of the symplectic gravity.

Let us restrict ourselves by the next-to-ordinary space-time case with $l = 2$. Three generic inequivalent types of the spinor decomposition relative to the embedding $Sp(4, C) \supset Sp(2, C)$ are conceivable: (i) $\underline{4} = \underline{2} \oplus \underline{2}$, (ii) $\underline{4} = \underline{2} \oplus \overline{2}$ and (iii) $\underline{4} = \underline{2} \oplus \underline{1} \oplus \underline{1}$.

(i) Chiral spinor doubling

$$\underline{4} = \underline{2} \oplus \underline{2} \tag{14}$$

results in the decomposition of the Hermitian second-rank spin-tensor $\underline{16} \sim \underline{4} \times \overline{4}$ as

$$\underline{16} = 4 \cdot \underline{4}\,, \tag{15}$$

i.e., in a collection of four four-vectors (more precisely, of three vectors and one axial vector, as follows from (12) and (13)). As for matter fermions, according to (14) the number of the two-component fermions after compactification is twice that of the number of the four-component fermions prior compactification. If a kind of the family structure reproduces itself during the compactification, it is necessary that there should be at least two copies of the fermions in the extended space-time with at least four copies of them in the ordinary space-time. For phenomenological reasons, the fermions in excess of three families should acquire rather large effective Yukawa couplings as a manifestation of the curled-up space-time background. This is not in principle impossible because the two-component fermions in (14) distinguish extra dimensions. Note, that the requirement for the renormalization group consistency of the Standard Model (SM) disfavours the fourth heavy chiral family in the model without a rather low cut-off [7]. But if due to the decomposition (15) for the gauge bosons, there appeared the additional moderately heavy vector bosons with the mass comparable to that of the heavy fermions, this constraint could in principle be evaded and the compactification scale Λ could be envisaged to be both rather moderate and high without conflict with the SM consistency. On the other hand, the extra time-like dimensions violate causality and the proper compactification scale Λ in the pseudo-orthogonal case is stated to be not less than the Planck scale [8]. Nevertheless, one may hope that the latter restriction could somehow be abandoned in the symplectic approach due to approximate causality here. It is to be valid at small boosts or gravitational fields, so that the compactification scale Λ could possibly be admitted to be not very high. For this reason, the given compactification scenario could still survive at any Λ.

(ii) Vector-like spinor doubling

$$\underline{4} = \underline{2} \oplus \overline{2} \tag{16}$$

results in the decomposition

$$\underline{16} = 2 \cdot \underline{4} \oplus \Big(\underline{3} + \text{h.c.}\Big) \oplus 2 \cdot \underline{1}\,. \tag{17}$$

In the traditional four-vector notations one has $X \sim (x_\mu^{(1,2)},\ x_{[\mu\nu]},\ x^{(1,2)})$, $\mu, \nu = 0, \ldots, 3$, with the tensor $x_{[\mu\nu]}$ being antisymmetric and all the components x being real. According to (16), after compactification there should emerge the pairs of the ordinary and mirror matter fermions. For phenomenological reasons, one should require the mirror fermions to have masses supposedly of the order of the compactification scale Λ. Modulo reservations for the preceding case, this compactification scenario could be valid at any Λ, too.

(iii) Spinor-scalar content

$$\underline{4} = \underline{2} \oplus \underline{1} \oplus \underline{1} \tag{18}$$

results in

$$\underline{16} = \underline{4} \oplus \Big(2 \cdot \underline{2} + \text{h.c.}\Big) \oplus 4 \cdot \underline{1}\,, \tag{19}$$

or in the mixed four-vector and spinor notations $X \sim (x_\mu,\ x_A^{(1,2)},\ x^{(1,2,3,4)})$, $A = 1, 2$. Due to (18), there would take place the violation of the spin-statistics connection for matter fields in the four-dimensional space-time if this connection fulfilled in the extended space-time. The scale of this violation should be determined by the compactification scale Λ which, in contrast with the two preceding cases, have safely to be high enough for not to violate causality within the experimental precision.

8 Gauge interactions

Let ${D_A}^{\bar B} \equiv {\partial_A}^{\bar B} + ig{G_A}^{\bar B}$ be the generic covariant derivative, with g being the gauge coupling, the Hermitian spin-tensor ${G_A}^{\bar B}$ being the gauge fields and ${\partial_A}^{\bar B} \equiv \partial/\partial {X^A}_{\bar B}$ being the ordinary derivative. Now let us introduce the strength tensor[5]

$$\begin{aligned} F^{[\bar B_1 \bar B_2]}_{\{A_1 A_2\}} &\equiv \frac{1}{ig} D^{[\bar B_1}_{\{A_1} D^{\bar B_1]}_{A_2\}} \\ &= \frac{1}{4ig}\Big(D^{\bar B_1}_{A_1} D^{\bar B_2}_{A_2} - D^{\bar B_2}_{A_2} D^{\bar B_1}_{A_1} + D^{\bar B_1}_{A_2} D^{\bar B_2}_{A_1} - D^{\bar B_2}_{A_1} D^{\bar B_1}_{A_2}\Big) \end{aligned} \tag{20}$$

[5]For simplicity, we do not distinguish in what follows the relative column positions of the indices of different kinds.

and similarly for $\overline{F}{}^{\{\bar{B}_1\bar{B}_2\}}_{[A_1A_2]} \equiv (F^{[\bar{A}_2\bar{A}_1]}_{\{B_2B_1\}})^*$, where $\{\ldots\}$ and $[\ldots]$ mean the symmetrization and antisymmetrization, respectively. One gets

$$F^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}} = \partial^{[\bar{B}_1}_{\{A_1} G^{\bar{B}_2]}_{A_2\}} + igG^{[\bar{B}_1}_{\{A_1} G^{\bar{B}_2]}_{A_2\}} \tag{21}$$

and similarly for $\overline{F}{}^{\{\bar{B}_1\bar{B}_2\}}_{[A_1A_2]}$. These tensors are clearly gauge invariant. The total number of the real components in the tensor $F^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}}$ is $2\cdot n(n-1)/2 \cdot n(n+1)/2 = n^2(n^2-1)/2$, and it exactly coincides with the number of components of the antisymmetric second-rank tensor $F_{[\alpha\beta]}$, $\alpha,\beta = 0,1,\ldots,n^2-1$, defined in the pseudo-Euclidean space of the $d=n^2$ dimensions. But in the symplectic case, tensor F is reducible and splits into a trace relative to ϵ and a traceless part, $F = F^{(0)} + F^{(1)}$, where $F^{(0)}{}^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}} \equiv F^{(0)}_{\{A_1A_2\}}\epsilon^{\bar{B}_1\bar{B}_2}$ and $F^{(1)}{}^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}}\epsilon_{\bar{B}_1\bar{B}_2} = 0$ (and similarly for $\overline{F}{}^{\{\bar{B}_1\bar{B}_2\}}_{[A_1A_2]}$). Hence, one has two independent irreducible representations with the real dimensionalities $d_0 = n(n+1)$ and $d_1 = n(n-2)(n+1)^2/2$. At $n=4$, one has in terms of the complex tensors of $SO(5,C)$: $F^{(0)}_{[IJ]} \equiv (\Sigma_{IJ})^{A_1A_2}F^{(0)}_{\{A_1A_2\}}$ and $F^{(1)}{}^{[\bar{B}_1\bar{B}_2]}_{[IJ]} \equiv (\Sigma_{IJ})^{A_1A_2}F^{(1)}{}^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}}$. At $n=2$, in terms of $SO(3,C)$ there remains only $F^{(0)}_{[ij]} \equiv (\sigma_{ij})^{A_1A_2}F^{(0)}_{\{A_1A_2\}}$ or, equivalently, $F^{(0)}_i \equiv 1/2\,\epsilon_{ijk}F^{(0)}_{[jk]}$.

For an unbroken gauge theory with fermions, the generic gauge, fermion and mass terms of the Lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_M$ are, respectively,

$$\begin{aligned} \mathcal{L}_G &= \sum_{s=0,1} (c_s + i\theta_s)\, F^{(s)}F^{(s)} + \text{h.c.}\,, \\ \mathcal{L}_F &= \frac{i}{2}\sum_{\pm}(\psi^\pm)^\dagger \overleftrightarrow{D}\,\psi^\pm\,, \\ \mathcal{L}_M &= \psi^+ m_0\,\psi^- + \sum_\pm \psi^\pm m_\pm \psi^\pm + \text{h.c.}\,, \end{aligned} \tag{22}$$

where $F^{(s)}F^{(s)} \equiv F^{(s)}{}^{[\bar{B}_1\bar{B}_2]}_{\{A_1A_2\}}F^{(s)}{}^{\{A_2A_1\}}_{[\bar{B}_2\bar{B}_1]}$. In the Lagrangian, m_0 is the generic Dirac mass, $m_\pm$ are Majorana masses, c_s and θ_s are the real gauge parameters. One of the parameters c_s, supposedly $c_0 \neq 0$, can be normalized at will. Eq. (22) results in the following generalization of the Dirac equation

$$iD^C{}_{\bar{B}}\psi^\pm_C = m_0^\dagger \overline{\psi}{}^\pm_{\bar{B}} + \sum_\pm m_\pm^\dagger\, \overline{\psi}{}^\mp_{\bar{B}} \tag{23}$$

and the pair of Maxwell equations ($c_0 \equiv 1$ and $c_1 = \theta_1 = 0$, for simplicity)

$$\begin{aligned} (1+i\theta_0)D^{C\bar{B}}F^{(0)}{}_{\{CA\}} - \text{h.c.} &= 0\,, \\ (1+i\theta_0)D^{C\bar{B}}F^{(0)}{}_{\{CA\}} + \text{h.c.} &= 2gJ_A{}^{\bar{B}}\,, \end{aligned} \tag{24}$$

with the fermion Hermitian current J given by (8).

The tensors $F^{(s)}$, $s = 1, 2$ are non-Hermitian, but under restriction by the maximal compact subgroup $Sp(2l)$ (when there is no distinction between the indices of different kinds) they split into a pair of the Hermitian ones $E^{(s)}$ and $B^{(s)}$ as follows: $F^{(s)} = E^{(s)} + iB^{(s)}$. Here one has $E^{(s)}{}^{[Y_1Y_2]}_{\{X_1X_2\}} \equiv 1/2[F^{(s)}{}^{[Y_1Y_2]}_{\{X_1X_2\}} + (F^{(s)}{}^{\{X_2X_1\}}_{[Y_2Y_1]})^*]$ and $B^{(s)}{}^{[Y_1Y_2]}_{\{X_1X_2\}} \equiv 1/2i[F^{(s)}{}^{[Y_1Y_2]}_{\{X_1X_2\}} - (F^{(s)}{}^{\{X_2X_1\}}_{[Y_2Y_1]})^*]$, so that $E^{(s)}{}^{[Y_1Y_2]}_{\{X_1X_2\}} = (E^{(s)}{}^{\{X_2X_1\}}_{[Y_2Y_1]})^*$ and similarly for $B^{(s)}$. Introducing the duality transformation $F^{(s)} \to \tilde{F}^{(s)} \equiv -iF^{(s)}$ with $\tilde{E}^{(s)} = B^{(s)}$ and $\tilde{B}^{(s)} = -E^{(s)}$, one gets $\mathcal{R}eF^{(s)}F^{(s)} = E^{(s)2} - B^{(s)2}$ and $\mathcal{I}mF^{(s)}F^{(s)} = \mathcal{R}e\tilde{F}^{(s)}F^{(s)} = 2E^{(s)}B^{(s)}$. Though the splitting into $E^{(s)}$ and $B^{(s)}$ is noncovariant with respect to the whole $Sp(2l, C)$, the duality transformation is covariant. The tensors $E^{(s)}$ and $B^{(s)}$ are the counterparts of the ordinary electric and magnetic strengths, and θ_0 is the counterpart of the ordinary T-violating θ-parameter for the $n = 2$ case. Thus, θ_1 is an additional T-violating parameter at $n > 2$. Note that in the framework of symplectic extension the electric and magnetic strengths stay on equal footing. This is to be compared with the pseudo-orthogonal extension where these strengths have unequal numbers of components at $d \neq 4$, and hence there is no natural duality relation between them. The electric-magnetic duality of the gauge fields (for imaginary time) play an important role for the study of the topological structure of the gauge vacuum in four space-time dimensions. Therefore, the similar study might be applicable to the case of the extended symplectic space-times with arbitrary $l > 1$.

The field equations (23) and (24) are valid in the flat extended space-time or, otherwise, refer to the inertial local frames. To go beyond, one can introduce the Hermitian local frames $e_{\alpha A}{}^{\bar{B}}(X)$, $e_{\alpha A}{}^{\bar{B}} = (e_{\alpha B}{}^{\bar{A}})^*$, with $\alpha = 0, 1, \ldots, n^2 - 1$ being the world vector index, the real world coordinates $x_\alpha \equiv e_\alpha{}^A{}_{\bar{B}} X_A{}^{\bar{B}}$, as well as the generally covariant derivative $\nabla_\alpha(e)$. Now, (22) can be adapted to the $d = n^2$ dimensional curved space-time equipped with a pseudo-Riemannian structure (the real symmetric metric $g_{\alpha\beta}(x) = e_\alpha{}^A{}_{\bar{B}} e_{\beta A}{}^{\bar{B}}$), or to the curved coordinates. In line with [9], one can also supplement gauge equations by the generalized gravity equations in the curved symplectic space-time. But now the group of equivalence of the local frames (structure group) is not the whole pseudo-orthogonal group $SO(d_-, d_+)$ but only its part isomorphic to $Sp(2l, C)$. It leaves more independent components in the local symplectic frames compared to the pseudo-Rimannian frames. The number of components in the latter ones being equal to that in the metrics, the symplectic gravity is not in general equivalent to the metric one. The curvature tensor in the symplectic case, like the gauge one, splits additionally into irreducible parts which can a priori enter the gravity Lagrangian with the independent coefficients. The ultimate reason for this may be that in the symplectic approach the space-time is likely to be not a fundamental

entity. By this token, gravity as a generally covariant theory of the space-time distortions is to be meant just as an effective theory. The latter admits the existence of a number of free parameters, the choice of which should be determined, in principle, by the physical contents of the effective theory and should ultimately be clarified by an underlying theory.

9 Conclusion

The hypothesis that the symplectic structure of space-time is superior to the metric one provides, in particular, the rationale for the four-dimensionality and $1+3$ decomposition of the ordinary space-time. When looking for the extra dimensional space-time extensions, the hypothesis predicts the discrete sequence of the metric space-times of the fixed dimensionalities and signatures. The symplectic extension proves to be not a priory inconsistent and provides a viable alternative to the pseudo-orthogonal one. The emerging dynamics in the extended space-time is largely unorthodox and possesses a lot of new features. The physical contents of the scheme require further investigation. But beyond the physical adequacy of the extra dimensional space-times, by generalizing from the basic case $l = 1$ to its counterpart for general $l > 1$, a deeper insight into the nature of the four-dimensional space-time itself may be attained.

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